

COMPRESSIVE SENSING FOR RADAR SIGNALS:

PART I: PROBABILISTIC APPROACH

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I. INTRODUCTION

Compressive sensing (CS) has received considerable research interest recently. CS has been successfully applied in many applications, such as high-resolution image processing, wireless communication, electronic signal capturing and biomedical imaging systems [1]. In the context of radar, the number of potential targets is often limited. Hence, exploiting the sparsity of radar signals in various spaces, CS has been recently applied to multiple-input multiple-output (MIMO) radar systems.

In principle, CS focuses on the role of sparsity in reducing the number of measurements needed to represent a finite dimensional vector. Since recovery from compressed measurements is an NP-hard problem, many suboptimal methods have been proposed to approximate the sparse solution. These algorithms recover the true value of the sparse vector when the columns of sparsity basis matrix are incoherent, called mutually incoherency (MC) criteria [7], [14], [8], [9].

A related criteria, referred to as restricted isometric property (RIP), is also used to guarantee the accuracy of sparse signal recovery [2], [3], [15], [11], [9], [16]. However, in practice, it is difficult to verify incoherency or RIP criteria. It is well known that a random measurement matrix typically satisfies the incoherency condition in most cases. Therefore, in related published works, the elements of the matrix Φ are randomly selected from a Gaussian distribution or a random pseudo random sequence $\{ \otimes 1 \}$.

However, it is difficult to determine the probability of perfect recovery in the context of CS theory. Recently, some efforts have been made to find conditions and/or tight bounds on the probability of perfect recovery (see [4], [5], [6] and the references therein). Although the nature of mutual coherence is random, the existing CS methods consider MC or RIP as constants when deriving the probability of perfect recovery [8], [9], [15], [16].

In this work, we provide a probabilistic approach for the problem of perfect sparse signal recovery under different incoherency conditions. To the best of our knowledge, this is the first paper that models both MC and RIP conditions as the combination of numerous independent and dependent

random variables (RVs). In this contribution, our ultimate goal is to find the perfect reconstruction probability for CS-based MIMO radars. Our results along with the developed conditions in [16] and [9] can be used to estimate the maximum number of detectable targets.

The rest of this paper is organized as follows. In Section II, we first present the signal model. Then, we discuss about the required conditions for the perfect signal reconstruction under both MC and RIP criteria. In Section III, we first define the probability of perfect recovery. Next, we derive the probability of perfect recovery in the sense of both mentioned conditions. Numerical examples are presented in Section IV.

Notation: Throughout this paper bold uppercase letters stand for matrices and bold lowercase letters stand for column vectors. $E[\cdot]$ is the statistical expectation and \mathbf{I} denote an identity matrix. In addition, $(\mathbf{A})_{N \times M}$ denote a matrix with N rows and M columns and $\{\cdot\}$ represents the real part of a complex quantity. Further, $\|\mathbf{a}\|_2$ denote the l_2 norm of \mathbf{a} and $\|\cdot\|$ represents the absolute value. Moreover, $\langle \cdot, \cdot \rangle$ introduce the inner product, $(\cdot)^T$ denote the transpose operator, and $(\cdot)^H$ stands for the Hermitian transpose.

II. CS-BASED MIMO RADAR

A. Signal Model

In this section, we describe the signal model for our MIMO radar system. We assume that there are M_t transmitters, M_r receivers, and K targets. Further, we assume that each of the targets contains Q individual isotropic scatterers. Let $\tilde{r}_{lp}^{k,q}$ denote the transmitted signal from l -th transmitter, reflected by the q th scatter on the k -th target, and received by the p -th receiver. Such a signal can be written as:

$$\tilde{r}_{lp}^{k,q} = \sqrt{\frac{E}{M_t}} \beta_q^k \{w_l(t - \tau_{lp}(x_k)) \exp(2\pi j(f_c(t - \tau_{lp}(x_k)) - f_{lp}^k(t - \tau_{lp}(x_k))))\}, \quad (1)$$

where $w_l(\cdot)$ denote the transmitted waveform, E represents the average energy of the transmitted signal, and f_c is the carrier frequency. Moreover, $\tau_{lp}(\cdot)$ and $f_{lp}^k(\cdot)$ are delay and Doppler shift

corresponding to the k th target, respectively. Note that $\{ \} w_l(t - \tau_{lp}(x_k)) \exp(2\pi \overline{1} f_c t)$ stands for the passband signal, β_q^k denote the reflectivity of the q -th scatter on the k -th target, and x_k is defined as the coordinator of the k -th target on the range axis.

Equation (1) can be written as:

$$\tilde{r}_{lp}^k = \sqrt{\frac{E}{M_t}} \delta_Q^k \{ \} w_l(t - \tau_{lp}(x_k)) \exp(2\pi \overline{1} f_c(t - \tau_{lp}(x_k))) | \exp(2\pi \overline{1} f_{lp}^k(t - \tau_{lp}(x_k))), \quad (2)$$

where $\delta_Q^k = \int_{q=1}^Q \beta_q^k$ is the total reflectivity of the k -th target. Considering the narrow-band assumption, we ignore the time delay and rewrite (2) as [12], [13]:

$$\tilde{r}_p(t) = \prod_{k=1}^K \sqrt{\frac{E}{M_t}} \delta_Q^k \prod_{l=1}^{M_t} \{ \} w_l(t) \exp(2\pi \overline{1} f_c(t - \tau_{lp}(x_k))) | \exp(2\pi \overline{1} f_{lp}^k(t - \tau_{lp}(x_k))) \quad (3)$$

The baseband representation for $\tilde{r}_p(t)$ is:

$$r_p(t) = \prod_{k=1}^K \sqrt{\frac{E}{M_t}} \delta_Q^k \prod_{l=1}^{M_t} w_l(t) \exp(-2\pi \overline{1} f_c \tau_{lp}(x_k)) \exp(2\pi \overline{1} f_{lp}^k(t - \tau_{lp}(x_k))). \quad (4)$$

Next, we transform (4) to discrete domain as:

$$r_p(n) = \prod_{k=1}^K \sqrt{\frac{E}{M_t}} \delta_Q^k \prod_{l=1}^{M_t} w_l(nT_s) \exp(-2\pi \overline{1} f_c \tau_{lp}(x_k)) \exp(2\pi \overline{1} f_{lp}^k(nT_s - \tau_{lp}(x_k))) \quad (5)$$

We discretize the range-Doppler space on a $M \pm L$ grid and rewrite (5) in the CS framework as:

$$r_p(n) = (\mathbf{c}_n)_{1 \times M_t} (\mathbf{A}_n)_{M_t \times ML} (\mathbf{s})_{ML \times 1}, \quad (6)$$

where $\mathbf{c}_n = [w_1(nT_s), w_2(nT_s), \dots, w_{M_t}(nT_s)]$, $\mathbf{s} = \left[\sqrt{\frac{E}{M_t}} \delta_Q^1 \times \dots \times \sqrt{\frac{E}{M_t}} \delta_Q^{ML} \right]^T$, and the elements of \mathbf{A}_n can be expressed as $(a_{i,k})_n = e^{-2\pi \sqrt{-1} f_c \tau_{M_t p}(x_{ML})} e^{2\pi \sqrt{-1} \int_{M_t p}^{ML} (nT_s - \tau_{M_t p}(x_{ML}))}$ for $i = 1, \dots, M_t, k = 1, 2, \dots, ML$.

Equation (6) can be expressed as:

$$\mathbf{r}_p = \begin{bmatrix} \mathbf{c}_1 & \mathbf{0} & \mathbf{A}_1 \\ & \ddots & \mathbf{A}_2 \\ \mathbf{0} & \mathbf{c}_N & \vdots \\ & & \mathbf{A}_N \end{bmatrix} \begin{bmatrix} \mathbf{s} \\ \mathbf{s} \\ \mathbf{s} \\ \mathbf{s} \end{bmatrix} = \underbrace{\mathbf{C} \mathbf{A}}_{\Psi} \mathbf{s} \quad (7)$$

Applying the measurement matrix, we obtain:

$$y(n) = \phi_m(\mathbf{r}_p)_n = \phi_m[\mathbf{0}, \times \times \times, \mathbf{C}_n, \times \times \times, \mathbf{0}]_{N \times M_t} (\mathbf{A}_n)_{M_t \times ML} (\mathbf{s})_{ML \times 1}, \quad (8)$$

where ϕ_m ($m = 1, \dots, M_m$) denotes the m -th row of the matrix Φ . We can rewrite (8) in matrix form as:

$$\mathbf{y} = \Phi \mathbf{r}_p = \Phi \Psi (\mathbf{s})_{ML \times 1}, \quad (9)$$

where Φ is the measurement matrix of size $M_m \times N$, $\Psi = (\mathbf{C})_{N \times NM_t} (\mathbf{A})_{NM_t \times ML}$ represents the basis matrix, $\Theta = \Phi \Psi$ denotes the sensing matrix of size $M_m \times ML$, and \mathbf{s} is a K -sparse vector.

B. Conditions on Mutual Coherence

MC of sensing matrix Θ is defined as:

$$\mu(\Theta) = \max_{i \neq j} \frac{\|\langle \theta_i, \theta_j \rangle\|}{\sqrt{\theta_i} \sqrt{\theta_j} \sqrt{\theta_i} \sqrt{\theta_j}}, \quad (10)$$

where θ_i and θ_j are different columns of matrix Θ . We aim to find the probability of perfect recovery in terms of $\mu(\Theta)$.

If the following condition is satisfied

$$K < 0.5 \left(1 + \frac{1}{\mu(\Theta)} \right), \quad (11)$$

then it is guaranteed that the OMP algorithm obtains the true \mathbf{s} . As stated in [16], (11) is a sufficient and necessary condition for the sparse signal recovery.

C. Conditions on RIP

Suppose that for every K -sparse vector \mathbf{v}_K , the constant δ exists such that

$$(1 - \delta) \|\mathbf{y}_K\|_2^2 \sim \|\Theta \mathbf{v}_K\|_2^2 \sim (1 + \delta) \|\mathbf{y}_K\|_2^2. \quad (12)$$

Let the isometry constant δ_K is defined as the minimum of all δ values that satisfying (12). As proved in [15], if RIP property is hold for $\delta_K < \frac{1}{3\sqrt{K}}$, then it is guaranteed that the OMP algorithm converges to the true s . Note that RIP is the sufficient condition for perfect signal recovery.

III. PERFECT RECOVERY PROBABILITY

A. Recovery Probability in MC sense

Defination: The probability of perfect recovery in the MC sense is defined as:

$$P^{MC} = \Pr \left(K < 0.5 \right) 1 + \frac{1}{\mu(\Theta)} \left[\left[\cdot \right] \right]. \quad \blacksquare \quad (13)$$

Based on the definition of Ψ , we have:

$$\psi_{uv} = \prod_{l=1}^{M_t} W_l(uT_s) \exp(-2\pi \overline{1} f_c \tau_{lq}(x_v)) \exp(2\pi \overline{1} f_{lq}^v(uT_s - \tau_{lq}(x_v))), \quad (14)$$

where x_v and f_{lq}^v are the range and Doppler points, respectively. As stated in (15), MC can be rewritten as the maximum of off-diagonal elements of the Gram matrix, i.e. $\mu(\Theta) = \max_{i \leq j} \|g_{ij}\|$.

In the context of CS, the basis matrix is typically pre-determined, while the measurement matrix is random. Therefore, the behavior of MC depends on Φ matrix. In general, elements of Φ are independently chosen from the random sequence “ $\{ \otimes 1 |$ ” or a Gaussian distribution. As we will explain, applying Gaussian distribution leads to high complexity.

Let us define the Gram matrix \mathbf{G} as follows:

$$\mathbf{G} = \Psi^H \Phi^H \Phi \Psi = [\mathbf{g}_1, \mathbf{g}_2, \dots, \mathbf{g}_N], \quad (15)$$

where $\mathbf{g}_j = [g_{1j} \times \times \times g_{ij} \times \times \times g_{Nj}]^T$ for $j = 1, 2, \dots, N$. Here, $g_{ij} = \int_k \psi_{kj} (\int_q (\psi_{qi} \int_p \phi_{pq}^H \phi_{pk}))$. To derive the probability of perfect recovery, we need to consider only the upper-right or lower-left half of the off-diagonal elements of \mathbf{G} . Note that g_{ij} 's are correlated RVs.

Here, we assume that \mathbf{g}_{off} contains the upper-right half of the off-diagonal elements of the Gram matrix. So, we have:

$$\frac{1}{N} \sqrt{\mathbf{g}_{\text{off}}} \sim \max_{i \neq j} \|g_{ij}\| \sim \sqrt{\mathbf{g}_{\text{off}}}. \quad (16)$$

The goal of our analysis is to quantify the probability of $\max_{i \neq j} \|g_{ij}\|$. For any $i \neq j$, $\|g_{ij}\|$'s are dependent RVs. To overcome this issue, we replace \mathbf{g}_{off} with the vector $\mathbf{g} = [g_1 \ g_2 \ g_3 \ \dots \ g_{N(N-1)/2}]^T$, where $\mathbf{g} \in \mathcal{N}(\boldsymbol{\mu}, \mathbf{C}_g)$. Note that $\sqrt{\mathbf{g}}^2 = (\sqrt{\int_{i=1}^{N(N-1)/2} \|g_i\|^2})^2 = \int_{i=1}^{N(N-1)/2} \|g_i\|^2$. Thus, applying eigenvalue decomposition, we have:

$$\mathbf{g} = \begin{bmatrix} \mathbf{U} & \mathbf{U}_0 \end{bmatrix} \sqrt{\begin{bmatrix} \boldsymbol{\Delta} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}} \begin{bmatrix} \mathbf{U}^H \\ \mathbf{U}_0^H \end{bmatrix}, \quad (17)$$

where

$$\begin{bmatrix} \mathbf{U} & \mathbf{U}_0 \end{bmatrix} \sqrt{\begin{bmatrix} \mathbf{U}^H \\ \mathbf{U}_0^H \end{bmatrix}} = \mathbf{I} \quad (18)$$

and

$$\begin{bmatrix} \mathbf{U}^H \\ \mathbf{U}_0^H \end{bmatrix} \begin{bmatrix} \mathbf{U} & \mathbf{U}_0 \end{bmatrix} = \mathbf{I}. \quad (19)$$

Thus, we have

$$\begin{bmatrix} \mathbf{U}^H \\ \mathbf{U}_0^H \end{bmatrix} \begin{bmatrix} \mathbf{U} & \mathbf{U}_0 \end{bmatrix} \sqrt{\begin{bmatrix} \boldsymbol{\Delta} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}} \begin{bmatrix} \mathbf{U}^H \\ \mathbf{U}_0^H \end{bmatrix} \begin{bmatrix} \mathbf{U} & \mathbf{U}_0 \end{bmatrix} = \begin{bmatrix} \boldsymbol{\Delta} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}. \quad (20)$$

Finally, after whitening process, we have:

$$\tilde{\mathbf{g}} = \begin{bmatrix} \mathbf{U}^H \\ \mathbf{U}_0^H \end{bmatrix} \mathbf{g}. \quad (21)$$

Note that $\frac{1}{\sqrt{N}} \|\tilde{\mathbf{g}}\|^2 = \left(\int_i \|g_i\|^2 \right) = \int_i \|g_i\|^2$. It is important to link the probability density function (PDF) of $\frac{1}{\sqrt{N}} \|\tilde{\mathbf{g}}\|^2$ with that one of $\max_i \|g_i\|^2$.

Lemma 1 Let us assume that the vector \mathbf{g} of size $n_R \pm 1$ is given by $\mathbf{h} = \overline{K_c \mathbf{h}} + \mathbf{h}_f$, where $\mathbf{h}_f \in \mathcal{N}_c(\mathbf{0}, \mathbf{R})$. Considering the fact that $E[(\mathbf{h} - \overline{K_c \mathbf{h}})(\mathbf{h} - \overline{K_c \mathbf{h}})^H] = E[\mathbf{h}_f \mathbf{h}_f^H] = \mathbf{R}$, we have $\mathbf{h} \in \mathcal{N}_c(\overline{K_c \mathbf{h}}, bR)$. Let $\lambda_1, \dots, \lambda_r$ be the nonzero eigenvalues of \mathbf{R} which means $\lambda_{r+1} = \dots = \lambda_{n_R} = 0$. The moment generating function (MGF) of $\frac{1}{\sqrt{N}} \|\mathbf{h}\|^2$ is obtained as:

$$M_{\|\mathbf{h}\|^2}(s) = \int_{j=1}^r \left[\frac{1}{1 - s\lambda_i} \exp \left(\frac{s\lambda_i K_c \|\mathbf{u}_i^H \bar{\mathbf{h}}\|^2}{1 - s\lambda_i} \right) \right] \left(\right), \quad (22)$$

where the eigenvector \mathbf{u}_i corresponds to the eigenvalue λ_i ($i = 1, 2, \dots, n_R$). ■

Proof: See the Appendix. ■

Now, equation (16) can be rewritten as:

$$\frac{1}{N} \|\mathbf{g}_{\text{off}}\|^2 \sim \max_{i \neq k} \|g_{ik}\|^2 \sim \frac{1}{N} \|\mathbf{g}_{\text{off}}\|^2. \quad (23)$$

Note that the MGF of RV X can be formulated as the Laplace transform of $f_X(a)$, i.e.

$$M_X(s) = \int_0^\infty f_X(x) \exp(-sx) dx. \quad (24)$$

Thus, the PDF of $X = \frac{1}{N} \|\mathbf{g}\|^2$ can be found through the following equation:

$$f_X(x) = \frac{1}{2\pi} \int_{c-\sqrt{-1}\infty}^{c+\sqrt{-1}\infty} \exp(sx) M_X(s) ds. \quad (25)$$

Lemma 2 The probability of perfect recovery P^{MC} is given by

$$P^{MC} = \Pr(\mu(\Theta) > \eta_{th}) = \frac{1}{2\pi} \int_{c-\sqrt{-1}\infty}^{c+\sqrt{-1}\infty} \frac{M_Z(s)}{s} \exp \left(\frac{s\eta_{th}^2}{\alpha^2} \right) ds, \quad (26)$$

where $\eta_{th} \triangleq 1/(2K - 1)$ denote the perfect recovery threshold $\mu(\Theta)$, and the MGF $M_Z(\cdot)$ is given in Lemma 1. ■

Proof: Let us define $Y = \overline{X}/\eta = \sqrt{\int \|g_i\|^2}/\eta$, where $\eta = N(N - 1)/2$. Then, the PDF of Y is given by:

$$f_Y(y) = \frac{\eta^2 f_x(y^2)}{\frac{1}{2y}} = \frac{\eta^2 y}{\pi} \int_{c-\sqrt{-1}\infty}^{c+\sqrt{-1}\infty} \exp(s\eta^2 y^2) M_X(s) ds. \quad (27)$$

Replacing $1/\eta$ by a curve fitting parameter α , results in:

$$\begin{aligned}
P^{MC} &= \frac{1}{\pi\alpha^2} \int_0^{\eta_{th}} \left[\int_{c-\sqrt{-1}\infty}^{c+\sqrt{-1}\infty} y \exp \right] \frac{sy^2}{\alpha^2} \left[M_X(s) ds dy \right. \\
&= \frac{1}{\pi\alpha^2} \int_{c-\sqrt{-1}\infty}^{c+\sqrt{-1}\infty} \left[\int_0^{\eta_{th}} y \exp \right] \frac{sy^2}{\alpha^2} \left[M_X(s) dy ds \right. \\
&= \frac{1}{2\pi} \int_{c-\sqrt{-1}\infty}^{c+\sqrt{-1}\infty} \left(\frac{M_Z(s)}{s} \right) \exp \left[\frac{s\eta_{th}^2}{\alpha^2} \right] ds.
\end{aligned} \tag{28}$$

This completes the proof. ■

We can use (26) to determine the perfect recovery probability. Note that only one integration is involved in the obtained solution.

B. Recovery Probability in RIP sense

In this subsection, we first define the probability of perfect recovery and then derive it as a function of δ and $\sqrt{y_K} \sqrt{\frac{2}{\delta}}$.

Definition The probability of perfect recovery in RIP sense is defined as:

$$P^{RIP}(\delta, \sqrt{y_K} \sqrt{\frac{2}{\delta}}) = \Pr(1 - \delta \sqrt{y_K} \sqrt{\frac{2}{\delta}} \sim U \sim 1 + \delta \sqrt{y_K} \sqrt{\frac{2}{\delta}}), \tag{29}$$

where δ is a constant, \mathbf{v}_K denote any K -sparse vector, and $U = \Theta \mathbf{v}_K \sqrt{\frac{2}{\delta}}$. ■

Here, we define $\Psi = [\psi_1, \psi_2, \dots, \psi_{ML}]$ and $\Phi = [\phi_1^T, \phi_2^T, \dots, \phi_{M_m}^T]^T$, where ϕ_i ($i = 1, 2, \dots, M_m$) denote the i -th row of matrix Φ .

Based on (12), we rewrite the middle term of inequality (14) as:

$$\Theta \mathbf{v}_K \sqrt{\frac{2}{\delta}} = \Phi \Psi \mathbf{v}_K \sqrt{\frac{2}{\delta}}, \tag{30}$$

where \mathbf{v}_K is a K -sparse vector, and $\kappa = \Phi \Psi \mathbf{v}_K = (\kappa_1, \dots, \kappa_i, \dots, \kappa_{M_m})^T$. In addition, $\kappa_i = \int_{p=1}^{ML} v_p \phi_i \psi_p$. As a result, we have $\Theta \mathbf{v}_K \sqrt{\frac{2}{\delta}} = \int_{i=1}^{M_m} \|\phi_i \mathbf{c}\|^2$, where \mathbf{c} stands for the constant vector $\mathbf{c} = \int_{p=1}^{ML} v_p \Psi_p$. Considering the independency property and recalling the center limited theorem, we conclude that $\Theta \mathbf{v}_K \sqrt{\frac{2}{\delta}}$ is a Gaussian RV. In what follows, we find the mean and variance of $\Theta \mathbf{v}_K \sqrt{\frac{2}{\delta}}$.

Since different elements of matrix Φ are independent and identically distributed Gaussian RVs with equal variance σ_ϕ^2 , the mean value of $\sqrt{\Phi \mathbf{v}_K} \sqrt{\frac{2}{\sqrt{2}}}$ can be expressed as:

$$E[\sqrt{\Phi \mathbf{v}_K} \sqrt{\frac{2}{\sqrt{2}}}] = E\left[\prod_{i=1}^{M_m} \|\phi_i \mathbf{c}\|^2 = \mathbf{c}^H\right) \prod_{i=1}^{M_m} \phi_i^H \phi_i \Big| \mathbf{c}. \quad (31)$$

The variance is obtained as:

$$\begin{aligned} Var[\sqrt{\Phi \mathbf{v}_K} \sqrt{\frac{2}{\sqrt{2}}}] &= E\left[\left(\prod_{i=1}^{M_m} \|\phi_i \mathbf{c}\|^2\right)^2\right] - (E[\sqrt{\Phi \mathbf{v}_K} \sqrt{\frac{2}{\sqrt{2}}}])^2 \\ &= E\left[\eta(\phi, \mathbf{c})^2 + 2\eta(\phi \leq \mathbf{c})\right] - (E[\sqrt{\Phi \mathbf{v}_K} \sqrt{\frac{2}{\sqrt{2}}}])^2 \\ &= \prod_{i=1}^{M_m} E[\|\phi_i \mathbf{c}\|^2] - (E[\sqrt{\Phi \mathbf{v}_K} \sqrt{\frac{2}{\sqrt{2}}}])^2 + E[\eta(\phi \leq \mathbf{c})]. \end{aligned} \quad (32)$$

In deriving (32), the following relation is used: $\int \prod_{i=1}^{M_m} \|\phi_{M_m} \mathbf{c}\|^2 \left(\prod_{i=1}^{M_m} \|\phi_i \mathbf{c}\|^2\right)^2 = \eta(\phi, \mathbf{c})^2 + 2\eta(\phi \leq \mathbf{c})$, where

$$\eta(\phi, \mathbf{c}) \triangleq \prod_{i=1}^{M_m} \|\phi_i \mathbf{c}\|^2 \quad (33)$$

and

$$\eta(\phi \leq \mathbf{c}) \triangleq \prod_{k=1}^{C_{M_m}^2} \|\phi_a \mathbf{c}\|^2 \|\phi_b \mathbf{c}\|^2. \quad (34)$$

The PDF of $\sqrt{\Phi \mathbf{v}_K} \sqrt{\frac{2}{\sqrt{2}}}$ is fully characterized by equations (31) and (32).

Lemma 3 The probability of perfect recovery $P^{RIP}(\delta, \sqrt{\frac{2}{\sqrt{2}}} \mathbf{y}_k)$ is given by:

$$P^{RIP}(\delta, \sqrt{\frac{2}{\sqrt{2}}} \mathbf{y}_k) = Q\left(\frac{1}{\sqrt{\sigma_u^2}}(\delta \sqrt{\frac{2}{\sqrt{2}}} \mathbf{y}_k + 2)\right) + Q\left(\frac{1}{\sqrt{\sigma_u^2}}(2 - \delta \sqrt{\frac{2}{\sqrt{2}}} \mathbf{y}_k)\right), \quad (35)$$

where σ_u^2 denote the variance of RV U , δ is a given constant in RIP definition, $Q(\cdot)$ is the Gaussian Q-function, and $\sqrt{\frac{2}{\sqrt{2}}} \mathbf{y}_K$ stands for an arbitrary K -sparse vector. In deriving (36), we used the relation $\text{erfi}(x) = \text{erfc}(-x)/\sqrt{\pi}$, where $\text{erfc} \triangleq 2 \int_0^x \exp(-t^2)/\sqrt{\pi} dt$. ■

Proof: The probability $P^{RIP}(\delta, \sqrt{y_k} \sqrt{\frac{2}{\delta}})$ obtains when matrix Θ satisfies the defined RIP condition in (12) for given vector \mathbf{v}_K and δ . Considering the PDF of $\sqrt{\Theta} \mathbf{v}_K \sqrt{\frac{2}{\delta}}$, we have:

$$\begin{aligned}
P^{RIP}(\delta, \sqrt{y_k} \sqrt{\frac{2}{\delta}}) &= \int_{(1-\delta)\|\mathbf{v}_K\|_2^2}^{(1+\delta)\|\mathbf{v}_K\|_2^2} f_U(u) du \\
&= \frac{1}{2\pi\sigma_u} \frac{1}{2} \sqrt{\frac{2}{2\pi\sigma_u^2}} \exp\left(-\frac{\mu^2}{4\sigma_u^4} \left[\operatorname{erfi}\left(\frac{1}{2\sigma_u^2} u + \frac{\mu}{2\sigma_u^2} \sqrt{2\sigma_u^2} \right) \right] \right) \Bigg|_{(1-\delta)\|\mathbf{v}_K\|_2^2}^{(1+\delta)\|\mathbf{v}_K\|_2^2} \\
&= Q\left(\frac{1}{\sqrt{\sigma_u^2}} (\delta \sqrt{y_k} \sqrt{\frac{2}{\delta}} + 2)\right) + Q\left(\frac{1}{\sqrt{\sigma_u^2}} (2 - \delta \sqrt{y_k} \sqrt{\frac{2}{\delta}})\right).
\end{aligned} \tag{36}$$

This completes the proof. \blacksquare

IV. NUMERICAL RESULTS

In this section, we present numerical results. The targets are assumed to fall on the grid points. Throughout our simulations, the carrier frequency is assumed to be 1 GHz, $M_t = M_m = 10$, and $K \in \{1, 3, 5\}$. In addition, the grid size is set to 105 ± 91 and 10 compressed measurements is used at each receiver.

Figure 1 shows the probability of perfect recovery in MC sense. Here, the elements of matrix Φ are drawn from a zero-mean Gaussian distribution. According to equation (11), radar is able to successfully detect $K = 6$ targets with the probability of 90%, while it can successfully detect $K = 5$ targets with the probability of 99.9%. In addition, decreasing the variance value leads to better performance. For example, changing σ_ϕ^2 from 0.7 to 0.1, improves the recovery probability from 90% to 96%.

In Fig. 2, we present the probability of perfect recovery in RIP sense. Here, we consider three different values for K . It is observed that increasing K improves the probability of perfect recovery. Also, it is seen that increasing the probability of perfect recovery leads to increase in δ value.

APPENDIX

PROOF OF LEMMA 1

We first apply a whitening process for the RV \mathbf{g} . Let $\tilde{\mathbf{g}}$ denote the outcome of the whitening process. Then, the mean and variance values of $\tilde{\mathbf{g}}$ can be written as:

$$\boldsymbol{\mu} = E[\tilde{\mathbf{g}}] = \begin{bmatrix} \mathbf{U}^H \\ \mathbf{U}_0^H \end{bmatrix} E[\mathbf{g}] = \begin{bmatrix} \mathbf{u}_1^H & \overline{K}\bar{\mathbf{g}} \\ \vdots & \vdots \\ \mathbf{u}_1^H & \overline{K}\bar{\mathbf{g}} \end{bmatrix} \quad (37)$$

and

$$Var(\tilde{\mathbf{g}}) = E[(\tilde{\mathbf{g}} - \boldsymbol{\mu})(\tilde{\mathbf{g}} - \boldsymbol{\mu})^H] = \begin{bmatrix} \boldsymbol{\Delta} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad (38)$$

where $\boldsymbol{\Delta}$ is a diagonal matrix. Further, $\boldsymbol{\mu} = (\mu_1, \mu_i, \mu_{n_R})$, where $\mu_i = \overline{K}\mathbf{u}_i^H \bar{\mathbf{g}}$. As a result, we have $\tilde{\mathbf{g}} \in N_c(\boldsymbol{\mu}, Var(\tilde{\mathbf{g}}))$. Note that all of the elements of $\tilde{\mathbf{g}}$ are independent RVs with distribution $h_i \in N_c(\mu_i, \lambda_i)$. Furthermore, we have $\|\tilde{\mathbf{g}}\|^2 = \sum_{i=1}^r \|\tilde{g}_i\|^2$, where $\|\tilde{g}_i\|^2$ s are Ricean RVs with the following PDF

$$f_{|\tilde{g}_i|}(x_i) = \frac{2x_i}{\lambda_i} \exp\left(-\frac{x_i^2 + \|\mu_i\|^2}{\lambda_i}\right) \left[I_0\left(\frac{2\|\mu_i\|x_i}{\lambda_i}\right)\right]. \quad (39)$$

Here, $I_0(\cdot)$ denote the modified Bessel function. So, the MGF of $\|\tilde{\mathbf{g}}\|^2$ is found as:

$$\begin{aligned} E[\exp(-s \sum_{i=1}^r \|\tilde{g}_i\|^2)] &= E\left[\int_0^\infty \exp(-sx_i^2) f_{|\tilde{g}_i|}(x_i) dx_i\right] \\ &= \int_0^\infty \exp(-sx_i^2) \left[\frac{2x_i}{\lambda_i} \exp\left(-\frac{x_i^2 + \|\mu_i\|^2}{\lambda_i}\right) \left[I_0\left(\frac{2\|\mu_i\|x_i}{\lambda_i}\right)\right] dx_i\right] \\ &= \int_0^\infty \exp(-sx_i^2) \left[\frac{1}{\lambda_i} \exp\left(-\frac{x_i^2 + \|\mu_i\|^2}{\lambda_i}\right) \left[I_0\left(\frac{2\|\mu_i\|x_i}{\lambda_i}\right)\right] dx_i\right] \\ &= \int_0^\infty \frac{1}{1-s\lambda_i} \exp\left(-\frac{y_i^2 + \|\mu_i\|^2}{\lambda_i}\right) \left[I_0\left(\frac{2\|\mu_i\|y_i}{\lambda_i}\right)\right] dy_i \\ &= \int_0^\infty \frac{1}{1-s\lambda_i} \exp\left(-\frac{y_i^2 + \|\mu_i\|^2}{\lambda_i}\right) \left[\frac{s\lambda_i \|\mu_i\|^2}{1-s\lambda_i}\right] dy_i \end{aligned} \quad (40)$$

This completes the proof.

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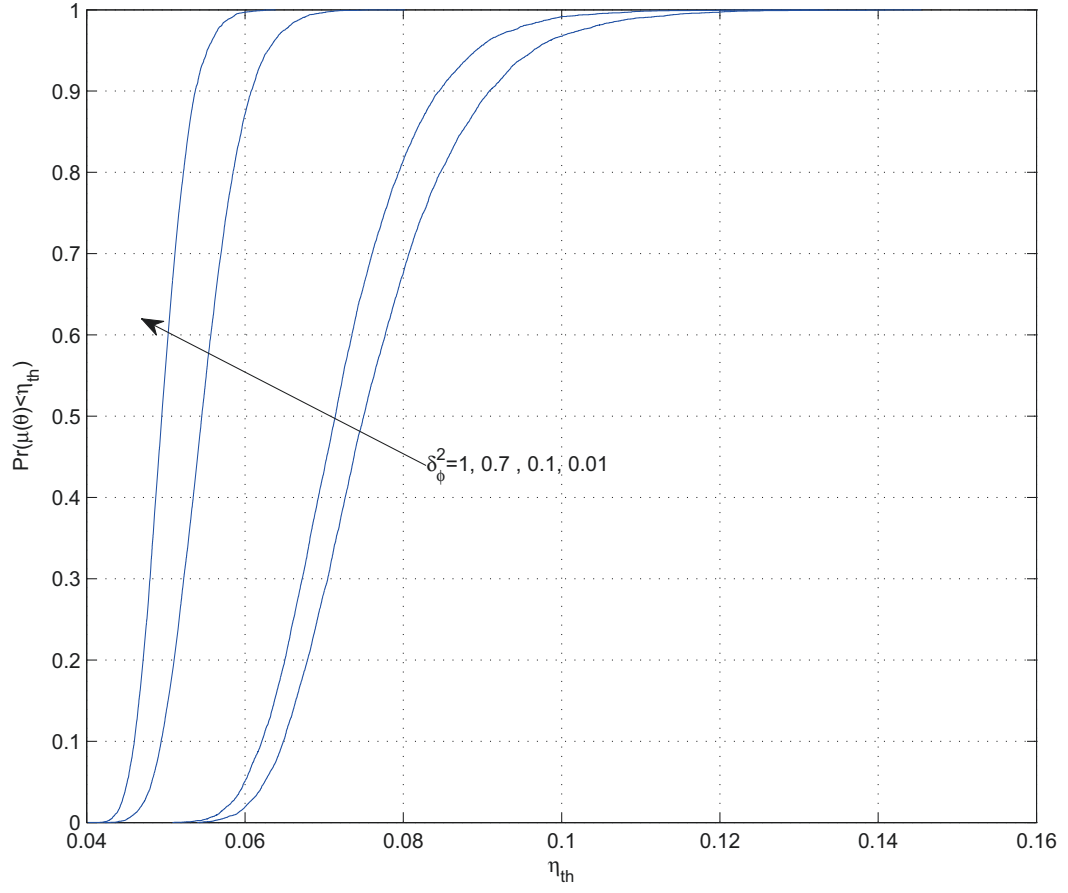


Fig. 1. Probability of perfect recovery in MC sense with different values of variance σ_ϕ^2 .

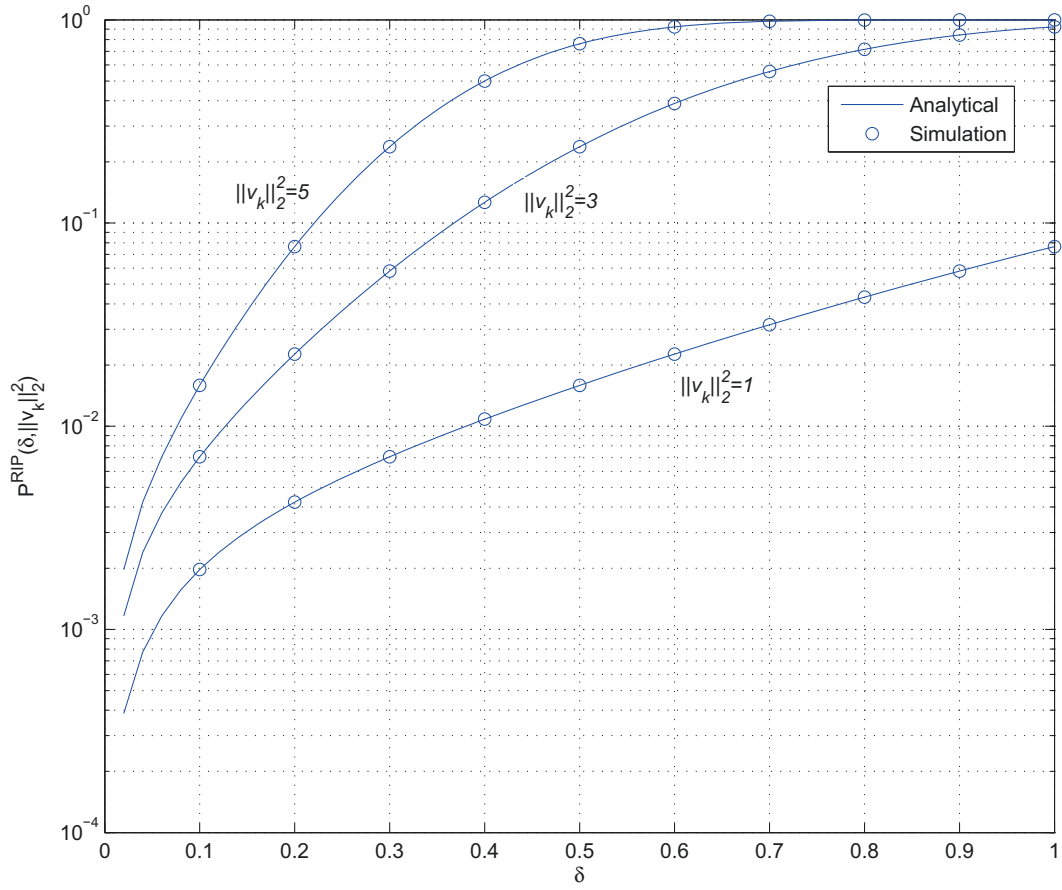


Fig. 2. Probability of perfect recovery in RIP sense for given K -sparse scenarios, where $K \in \{1, 3, 5\}$.